Nonreciprocal Propagation of Guided Waves in Sliding Laminae of Isotropic Dielectrics

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The well-known relation of Fresnel and Lorentz for the effect of material motion on the propagation of light is extended to guided waves propagating in a system of dielectric layers differing in refractive index as well as in the magnitude of their velocities. The analysis is based only on the existence of a transverse dispersion (determinantal equation) which is invariant with respect to sliding motion at least to first order in β. A one-to-one correspondence ("transmodification") of wave vectors in the moving and in the quiescent state is used to demonstrate that nonuniform motion can lead to (convectively) unstable modes. The theory is applied to a moving parallel-plate waveguide and to the model of a glow discharge enclosed in a circular waveguide.

Introduction

Whenever a wave is influenced by the velocity of the material through which it propagates we shall say that the wave suffers "dragging". The influence can affect the phase as well as the amplitude. Dragging is an inherently nonreciprocal phenomenon. In its general definition given here it is infact the only means known so far that can produce nonreciprocal propagation in the absence of (static) magnetisation. Excluding typical bunching effects as in electron beam devices from the following discussion, we want however to restrict ourselves to conditions where the effect of the medium on the wave can be described in its rest frame by scalars like ε, μ, which depend only on frequency and position.

In principle dragging allows to determine the state of motion of refringent material from the amount of non-reciprocity (phase and amplitude) observed in the propagation of two opposite waves. This possibility deserves interest for instance in the study of moving plasmas, although for magnetized plasma the analysis must be extended to the case of a dielectric tensor. We shall see below that dragging in general is most pronounced for slow waves – and this indeed seems quite natural – where $v_{\text{phase}} \ll c$ and for large dispersion of the moving material. Waves propagating near resonance in inhomogeneous and/or magnetized plasmas confined by conducting walls are of such kind. We shall also see that dragging by material in nonuniform motion can lead to wave amplification. This aspect however will not be treated in much detail.

It is evident that prior to solving the problem of dragging we have to know the dispersion of the system with all material at rest. For a guided wave this means the solution of a boundary value problem which is involved rather in practice than in principle and which has been solved quantitatively only for a few numbers of idealized cases. (For plasma filled waveguides a systematic survey is given in Reference [1].) We have tried to remove to the appendix the treatment of the boundary value problem needed for our purpose.

I. The Dragging of Guided Waves by Dielectrics in Uniform Motion

The "drag" exerted on a light wave which propagates freely and parallel to the material velocity v is given to first order in v/c by the well-known expression

$$v_{\text{ph}}(v) - v_{\text{ph}}(0) = v[1 - n^{-2} + (\omega/n)\, dn/d\omega] \tag{1}$$

of which the first two terms are due to Fresnel [2] and the last one has been added by Lorentz [3] to account for the dispersion of the medium. Since Eq. (1) reflects nothing more than the (relativistic) addition of two velocities with due correction for the Doppler shift, it will hold for all kinds of waves, not neccessarily electromagnetic in nature. In particular we can apply it to a guided wave of the form

$$f(x,y)\, \exp\{i(k_z z - \omega t)\} \tag{2}$$

provided we can find a rest frame for all material of influence. We have simply to replace n in Eq. (1) by the "index of mode refraction", N, according to the scheme

$$n \to N = k_z/k_0\,, \quad k_0 = \omega/c\,. \tag{3}$$

The dragging for guided waves by material in uniform motion is therefore once again given by:

$$\Delta v_{\mathrm{ph}} = v_{\mathrm{ph}}(v) - v_{\mathrm{ph}}(0) = v[1 - N^{-2} + (\omega/N)\,\mathrm{d}N/\mathrm{d}\omega] = v\,\varkappa \quad (4)$$

where $\varkappa$ is the "dragging coefficient".

Historically this quantity played a role in the speculations on the existence of a "light ether" during the last century. We want to replace $\varkappa$ for the rest of this paper by another coefficient δ defined by

$$-\Delta k_z/\beta\, k_0 = [k_z(0) - k_z(v)]/\beta\, k_0 = \delta = N^2\,\varkappa\,. \quad (5)$$

The use of δ instead of $\varkappa$ simplifies somewhat the calculation and allows a more direct interpretation of formulas in physical terms. For instance, the derivation of Eq. (4) in terms of δ becomes very simple if we start from the transformation of a wave vector (k_0', k_z') in a frame moving with velocity $v_z = v = \beta\, c$ in the laboratory. In the latter this very vector appears as (k_0, k_z) which is given by

$$k_0 = \gamma(k_0' + \beta\, k_z'); \quad k_z = \gamma(k_z' + \beta\, k_0'); \quad \gamma = (1-\beta^2)^{-1/2}.$$

Retaining only terms linear in β this reduces to

$$k_0 = k_0' + \beta\, k_z'; \quad k_z = k_z' + \beta\, k_0'. \quad (6)$$

Thus the mode (k_0', k_z') has the "frequency" $k_0' + \beta\, k_z'$ in the laboratory. In order to arrive at the frequency k_0' in the laboratory we must start from the mode

$$(k_0'' = k_0' - \beta\, k_z',\ k_z'' = k_z' - (\mathrm{d}k_z/\mathrm{d}k_0)\,\beta\, k_z')$$

in the moving frame where $\mathrm{d}k_z/\mathrm{d}k_0$ has to be taken from the dispersion curve in that frame (rest frame of the medium). Now the mode (k_0'', k_z'') appears as $(k_0', k_z'' + \beta\, k_0')$ in the laboratory and can be compared with (k_0', k_z'), the mode that results from Eqs. (6) for $\beta = 0$:

$$-\frac{\Delta k_z}{\beta\, k_0'} = -\frac{k_z'' + \beta\, k_0' - k_z'}{\beta\, k_0'} = \left[\frac{\mathrm{d}k_z}{\mathrm{d}k_0}\cdot\frac{k_z}{k_0}\right]_{\text{rest frame}} - 1$$
$$= \mathrm{d}k_z^2/\mathrm{d}k_0^2 - 1 = c^2/v_{\mathrm{gr}}\cdot v_{\mathrm{ph}} - 1 = \delta\,. \quad (7)$$

In order to calculate k_z by differentiation we also had to assume $\beta\, k_z\ (\mathrm{d}k_z/\mathrm{d}k_0) \ll k_z$ or $\beta \ll \mathrm{d}k_0/\mathrm{d}k_z = v_{\mathrm{gr}}/c$.

From the last line of Eq. (7) we can interpret the quantity δ on a (k_z^2, k_0^2)-diagram as has been done in Figure 1.

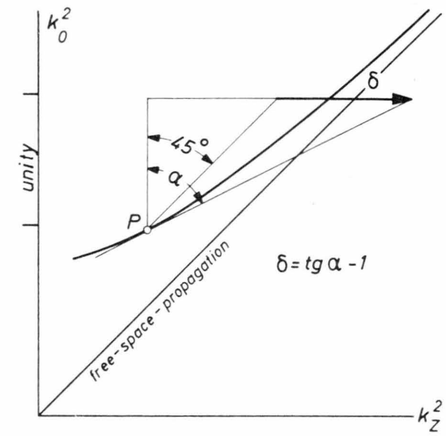

Fig. 1. Geometrical representation of the dragging coefficient δ.

Two different dielectrics, transverse dispersion

So far we have not made any assumption on the dielectric profile. Now we assume that the wave (2) propagates parallel to the interface of two regions of different refractive index n_1, n_2. This implies that the propagation vectors in the two regions have common components k_z, k_y parallel to the interface. We then have in the rest frame

$$n_1^2\, k_0^2 = k_z^2 + q_1^2; \quad q_1^2 = k_{y_1}^2 + k_{x_1}^2 \text{ in region 1}$$
$$(8)$$
$$n_2^2\, k_0^2 = k_z^2 + q_2^2; \quad q_2^2 = k_{y_2}^2 + k_{x_2}^2 \text{ in region 2}$$

Eqs. (8) are the dispersion relations for the individual media in their common rest frame, they allow to express k_0^2 and k_z^2 in terms of the transverse components q_1^2 and q_2^2:

$$k_0^2 = \frac{q_1^2 - q_2^2}{n_1^2 - n_2^2}\,; \quad k_z^2 = \frac{n_2\, q_1^2 - n_1\, q_2^2}{n_1^2 - n_2^2}\,; \quad (9)$$

For $n_1^2 \neq n_2^2$ any dispersion curve $D^{(0,z)}(k_0^2, k_z^2) = 0$ in the $k_0^2 - k_z^2$-plane corresponds to a dispersion curve $D^{(1,2)}(q_1^2, q_2^2) = 0$ in the (q_1^2, q_2^2)-plane, the laws of transformation given by Eq. (9) being linear in the squares of the wavenumbers.

For future application we still want to express δ in terms of transverse wavenumbers.

From Eqs. (7) and (8) we find

$$\delta = n_i^2 - 1 + k_0^2\,\frac{\mathrm{d}n_i^2}{\mathrm{d}k_0^2} - \frac{\mathrm{d}q_i^2}{\mathrm{d}k_0^2}\,; \quad i = 1, 2\,. \quad (10)$$

Whenever n_1^2 is of the form

$$n_1^2 = 1 - \mathrm{const}/k_0^2. \quad (11)$$

Equation (10) reduces to a particularly simple expression for δ. We then have

$$\delta = -\mathrm{d}q_1{}^2/\mathrm{d}k_0{}^2. \qquad (12)$$

Equation (12), for instance, gives the dragging coefficient for a partially filled waveguide ($n_1 = 1$). It also shows immediately that the empty waveguide with perfectly conducting walls (no field penetration) will have no dragging effect since $q_1 = q_2 =$ const. as a consequence of the boundary conditions. (For further discussions see Section VI below.)

II. Two Different Nondispersive Dielectrics Moving with Parallel Velocities of Individual Magnitude $v_{1,2} \ll c$

The situation differs from the one discussed so far because a coordinate system in which all refractive material comes to rest no longer exists. In Fig. 2 two dielectrics are moving at velocities v_1 and v_2 parallel to the z-axis in the laboratory. Wave

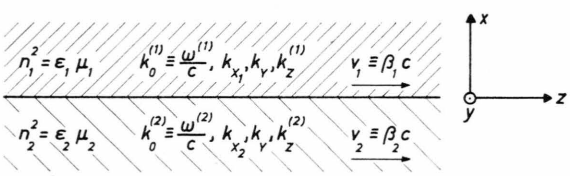

Fig. 2. Coordinates and wave numbers at the interface of two sliding dielectrics.

number components which refer to the rest frame are labelled by upper indices in brackets. The existence of a common wave in the two regions now still implies common y-, z- and 0-components of the propagation vector in the laboratory system but different z- and 0-components in the two rest systems. Equations (8) are now replaced by

$$n_i^2 k_0{}^{(i)2} = k_z{}^{(i)2} + q_i^2; \qquad i = 1, 2 \qquad (13)$$

where we could set $q^{(i)} = q_i$ since this quantity is not affected by the transformation. The transformation of the propagation vector from system (1) and (2) to the laboratory is given by

$$k_0 = \gamma_i (k_0{}^{(i)} + \beta_i k_z{}^{(i)}); \quad k_z = \gamma_i (k_z{}^{(i)} + \beta_i k_0{}^{(i)}) \qquad (14)$$

where $\qquad \gamma_i = (1 - \beta_i^2)^{-1/2}$.

Equations (14) describe the (relativistic) Doppler shift in frequency and wavenumber. Neglecting again terms of higher than first order in β_i, we replace (14) by

$$k_0 = k_0{}^{(i)} + \beta_i k_z{}^{(i)}; \quad k_z = k_z{}^{(i)} + \beta_i k_0{}^{(i)}, \qquad (15)$$

and the reverse relations

$$k_0{}^{(i)} = k_0 - \beta_i k_z; \quad k_z{}^{(i)} = k_z - \beta_i k_0 \qquad (16)$$

We shall call Eqs. (15) and (16) "semirelativistic Doppler transformations". They carry one characteristic element of special relativity which is the symmetry between space and time. They do not carry the other element which is the scaling law given by the Lorentz contraction factor γ^{-1}.

In Appendix F we have demonstrated that our boundary value problem is characterized by a determinantal equation $D^{(1,2)}(q_1, q_2)$ which is invariant in β_i at least to first order. Making use of this property we can find $D^{(0,z)}(k_0, k_z)$ from $D^{(1,2)}(q_1, q_2)$ simply by expressing k_0 and k_z in terms of q_1, q_2 and the velocities β_1, β_2. Assuming still $\beta_i \ll 1$ we write Eqs. (13) with the help of Eqs. (16) in terms of laboratory wave numbers.

$$n_i^2 k_0^2 - k_z^2 - 2(n_i^2 - 1)\beta_i k_0 k_z - q_i^2 = 0, \; i = 1, 2. \qquad (17)$$

Eliminating first k_z^2 then k_0^2 from Eqs. (17) yields the equivalent system

$$q^2 = n_{12}{}^2 k_0^2 - 2 p k_0 k_z \approx (n_{12} k_0 - (p/n_{12}) k_z)^2,$$
$$q\hat{}^2 = n_{12}{}^2 k_z^2 - 2 p\hat{} k_0 k_z \approx (n_{12} k_z - (p\hat{}/n_{12}) k_0)^2 \qquad (18)$$

where

$$n_{12}{}^2 = n_1{}^2 - n_2{}^2,$$
$$p = p_1 - p_2; \qquad p\hat{} = n_2{}^2 p_1 - n_1{}^2 p_2,$$
$$p_i = (n_i^2 - 1)\beta_i; \qquad i = 1, 2, \qquad (19)$$
$$q^2 = q_1{}^2 - q_2{}^2; \qquad q\hat{}^2 = n_2{}^2 q_1{}^2 - n_1{}^2 q_2{}^2,$$

and where the $\approx$ signs hold because the error is of the order of β_i^2. Solving for k_0, k_z we obtain (to first order in β_i) (20)

$$k_0 = \pm \frac{q}{n_{12}} \pm \frac{p}{n_{12}{}^2} \frac{q\hat{}}{n_{12}}; \quad k_z = \pm \frac{q\hat{}}{n_{12}} \pm \frac{p\hat{}}{n_{12}{}^2} \frac{q}{n_{12}}.$$

Eqs. (20) together with the definitions of Eqs. (19) give the desired wavevector components k_0, k_z for a mode characterized by the transverse components q_1, q_2 and the velocities β_1 and β_2. Depending on whether the same or the opposite sign is attributed to q and $\langle \hat{q} \rangle$ we obtain two different modes with antiparallel propagation, each of them affected differently by the streaming of the material. Reversing simultaneously the sign of p and $\langle \hat{p} \rangle$ (i.e. reversing the sign of β_1 as well as β_2) is equivalent to changing the relative sign of q and $\langle \hat{q} \rangle$.

Setting

$$q/n_{12} = k_{00}; \qquad q\hat{}/n_{12} = k_{z0} \qquad (21)$$

we can write Eqs. (20) as follows

$$k_0 = k_{00} + (p/n_{12}^2)\, k_{z0}\,; \qquad k_z = k_{z0} + (\hat{p}/n_{12}^2)\, k_{00}$$
$$= k_{00} + \mathrm{d}k_0^* \qquad\qquad = k_{z0} + \mathrm{d}k_z^* \qquad (22)$$

where, using the definitions (19),

$$\mathrm{d}k_0^* = \mathrm{d}k_{01}^* + \mathrm{d}k_{02}^*, \quad \mathrm{d}k_{0i}^* = (n_i^2 - 1)\, k_{z0}\, \beta_i / n_{ik}^2, \qquad (23)$$

$$\mathrm{d}k_z^* = \mathrm{d}k_{z_1}^* + \mathrm{d}k_{z_2}^*, \quad \mathrm{d}k_{zi}^* = n_k^2 (n_i^2 - 1)\, k_{00}\, \beta_i / n_{ik}^2, \qquad (24)$$

$$i = 1, 2; \quad k = 2, 1\,.$$

The ambiguity of the signs which appears in Eqs. (20) is removed from Eqs. (22) because for vanishing β_1, β_2 we have $k_0 \to k_{00}$, $k_z \to k_{z0}$.

For the calculation of Δk_z we now use arguments very similar to those which led to Equation (7). We start from a mode which propagates in the quiescent material and which has the wavenumbers

$$(k_{00} - \mathrm{d}k_0^*,\ k_{z\,0} - (\mathrm{d}k_{z\,0}/\mathrm{d}k_{00})\, \mathrm{d}k_0^*)\,.$$

The continuous transition to the moving state (β_1, β_2) causes a mode deformation (termed "transmodification" in Section V) which is governed by Eqs. (22) and which terminates in the mode

$$(k_{00}\,,\ k_{z\,0} - (\mathrm{d}k_{z\,0}/\mathrm{d}k_{00})\, \mathrm{d}k_0^* + \mathrm{d}k_z^*)\,.$$

We find

$$\Delta k_z = -\,(\mathrm{d}k_{z0}/\mathrm{d}k_{00})\, \mathrm{d}k_0^* + \mathrm{d}k_z^*$$
$$= -\,(\mathrm{d}k_{z0}/\mathrm{d}k_{00})\, \frac{p}{n_{12}^2}\, k_{z0} + \frac{\hat{p}}{n_{12}^2}\, k_{00}\,. \qquad (25)$$

After introduction of the definitions given in Eqs. (19) we can write the result in the form

$$-\,\Delta k_z / k_{00} = \delta_1 \beta_1 + \delta_2 \beta_2 \qquad (26)$$

with

$$\delta_1 = \frac{n_1^2 - 1}{n_1^2 - n_2^2} \left(\frac{\mathrm{d}k_z^2}{\mathrm{d}k_{00}^2} - n_2^2 \right) \qquad (27)$$

or likewise

$$\delta_1 = -\left(\frac{n_1^2 - 1}{n_1^2 - n_2^2} \right) \frac{\mathrm{d}q_2^2}{\mathrm{d}k_0^2} \qquad (28)$$

where the last version follows from Eqs. (8) and where δ_2 is obtained by permutation of subscripts.

Setting $\beta_1 = \beta_2 = \beta$ must necessarily lead back to the Fresnel-Lorentz relation of Eq. (7). The same must be true for $n_2^2 = 1$ for the reason that we can attribute to the vacuum any velocity we like.

Equations (27) and (28) are the extension of Eqs. (7) and (12) for non-uniform motion. It appears that they break down for $n_1^2 = n_2^2 \neq 1$. However this latter case can still yield a finite limit. For instance in boundary value problems with lossless dielectrics enclosed in perfectly conducting surfaces we know, that for $n_1 = n_2 = \mathrm{const}$, q_2 is determined only by the transverse dimensions (guiding surfaces). Therefore $\mathrm{d}q_2^2/\mathrm{d}k_0^2$ vanishes simultaneously with $n_1^2 - n_2^2$ and in order to find the limit we have to know more about the dispersion of the system (see the following section).

III. Two Dispersive Dielectrics

For simplicity we assume $\mathrm{d}\mu_i/\mathrm{d}k_0 = 0$ as the analysis for $\mathrm{d}\mu_i/\mathrm{d}k_0 \neq 0$ will only be a trivial extension. Assuming that the changes $\mathrm{d}\varepsilon_i$ introduced by the Doppler shifts as given by Eq. (15) remain small, we can correct k_z for these changes in very much the same way as we have corrected for the Doppler shift of k_z itself in the absence of material dispersion. However, to do this, we first have to consider all spatial wavevector components as dependent on the five variables k_0, ε_1, ε_2, β_1, β_2 i. e.

$$k_z = k_z(k_0\,, \varepsilon_1\,, \varepsilon_2\,, \beta_1\,, \beta_2)\,, \quad q_i = q_i(k_0\,, \varepsilon_1\,, \varepsilon_2\,, \beta_1\,, \beta_2) \qquad (29)$$

irrespective of the intrinsic dependence of ε_i on k_0. In the first step we proceed as just before, keeping ε_i fixed at the values

$$\varepsilon_i^{(i)} = \varepsilon_i(k_0^{(i)}) \qquad (30)$$

which they had in their respective rest frames. In the second step we note that k_z underdgoes a small change $\mathrm{d}k_z^{**}$ when the ε_i are shifted by small amounts $\mathrm{d}\varepsilon_i = (\mathrm{d}\varepsilon_i/\mathrm{d}k_0)\, \mathrm{d}k_{0i}$ given by

$$\mathrm{d}k_z^{**} = \sum_i \frac{\partial k_z}{\partial \varepsilon_i}\, \mathrm{d}\varepsilon_i = \sum_i \frac{\partial k_z}{\partial \varepsilon_i} \frac{\mathrm{d}\varepsilon_i}{\mathrm{d}k_0}\, \mathrm{d}k_{0i}\,, \qquad (31)$$

[Note that the $\mathrm{d}k_{0i} = \beta_i k_z \approx \beta_i k_{z0}$ in Eq. (31) are true Doppler shifts which have to be taken from Equations (15).] Grouping again into separate contributions from β_1 and β_2 we find that Eqs. (27), (28) have only to be supplemented by an additional term, i. e.

$$\delta_1 = \frac{n_1^2 - 1}{n_1^2 - n_2^2} \left(\frac{\partial k_z^2}{\partial k_0^2} - n_2^2 \right) + \frac{\partial k_z^2}{\partial \varepsilon_1} \frac{\mathrm{d}\varepsilon_1}{\mathrm{d}k_0^2} \qquad (32)$$

or alternatively

$$\delta_1 = -\,\frac{n_1^2 - 1}{n_1^2 - n_2^2} \frac{\partial q_2^2}{\partial k_0^2} + \frac{\partial k_z^2}{\partial \varepsilon_1} \frac{\mathrm{d}\varepsilon_1}{\mathrm{d}k_0^2} \qquad (33)$$

As before δ_2 is obtained by permutation of subscripts. Introduction of $\delta_{1,2}$ into Eq. (26) yields Δk_z.

Expressions for δ in terms of $D(q_1^2, q_2^2, \varepsilon_1, \varepsilon_2)$

For later application it is still desireable to express the derivatives $\partial k_z^2/\partial k_0^2$ and $\partial k_z^2/\partial \varepsilon_i$ occurring in Eqs. (32) and (33) in terms of derivatives of the function $D(q_1^2, q_2^2, \varepsilon_1, \varepsilon_2)$ defined in Appendix E. We remember that k_z ultimately always has to be calculated from this determinantal equation and Equations (8). For our purpose we use these equations in differential form. From the first of Eqs. (20A) we obtain

$$D_1\, dq_1^2 + D_2\, dq_2^2 + D_{\varepsilon_1}\, d\varepsilon_1 + D_{\varepsilon_2}\, d\varepsilon_2 = 0 \tag{34}$$

where

$$D_i = \partial D/\partial q_i^2;\; D_{\varepsilon_i} = \partial D/\partial \varepsilon_i\,.$$

(From the second equation we obtain the same after D is replaced by $\bar{D}$.)

Assuming again $\mu_i = 1$ or $n_i^2 = \varepsilon_i$ Eqs. (9) write in differential form

$$(\varepsilon_1 - \varepsilon_2)\, dk_0^2 + k_0^2\, d\varepsilon_1 - k_0^2\, d\varepsilon_2 = dq_1^2 - dq_2^2 \tag{35}$$

and

$$\begin{aligned}(\varepsilon_1 - \varepsilon_2)\, dk_z^2 + \varepsilon_2\, k_0^2\, d\varepsilon_1 - \varepsilon_1\, k_0^2\, d\varepsilon_2 \\ = \varepsilon_2\, dq_1^2 - \varepsilon_1\, dq_2^2.\end{aligned} \tag{36}$$

By elimination of dq_i^2 from Eqs. (34), (35) and (36) and equating to zero in turn two of the other differentials we obtain

$$\begin{aligned}\partial k_z^2/\partial k_0^2 &= (D_1\, \varepsilon_1 + D_2\, \varepsilon_2)/(D_1 + D_2), \\ \partial k_z^2/\partial \varepsilon_i &= (D_{\varepsilon_i} + k_0^2\, D_i)/(D_1 + D_2)\,;\; i = 1, 2\,.\end{aligned} \tag{37}$$

When Eqs. (37) are introduced into Eqs. (32), with $n_i^2 = \varepsilon_i$, assuming $\mu_i = 1$, we obtain

$$\begin{aligned}\delta_i = D_i\, \delta_{i\infty}/(D_1 + D_2) \\ + D_{\varepsilon_i}(d\varepsilon_i/dk_0^2)/(D_1 + D_2), \quad i = 1, 2\end{aligned} \tag{38}$$

where (39)

$$\delta_{i\infty} = \varepsilon_i - 1 + k_0^2(d\varepsilon_i/dk_0^2) = n_i^2 - 1 + k_0^2(dn_i^2/dk_0^2)$$

according to Eq. (1) [or Eq. (10) for $q_i = 0$], is the coefficient δ for free propagation of a plane wave in the unlimited medium. In particular if $\delta_{1\infty}$ happens to be zero (a corresponding case is treated in more detail in Section VII) we arrive at an expression which – after all – is simple enough in structure

$$\delta_{1\infty} = 0: \quad \delta_1 = D_{\varepsilon_1}(d\varepsilon_1/dk_0^2)/(D_1 + D_2)\,. \tag{40}$$

For the particular case $n_2^2 = \varepsilon_2 = 1$ it is easy to show that Eq. (40) gives the same result as Eq. (33): First, we find from the definition of $\delta_{1\infty}$ [Eq. (39)] that the condition $\delta_{1\infty} = 0$ imposes on $\varepsilon_1(k_0)$ the dependence

$$\varepsilon_1 = 1 - \text{const}/k_0^2. \tag{41}$$

Second, we find from Eq. (35) that due to this form

$$dq_1^2 = dq_2^2 \tag{42}$$

Finally, from Eq. (34) we obtain

$$d\varepsilon_1 = -(D_1 + D_2)\, dq_2^2/D_{\varepsilon_1}\,. \tag{43}$$

Setting

$$\begin{aligned}dq_2^2 &= \frac{\partial q_2^2}{\partial k_0^2}\, dk_0^2 + \frac{\partial q_2^2}{\partial \varepsilon_1}\, d\varepsilon_1 \\ &= \left(\frac{\partial q_2^2}{\partial k_0^2} - \frac{\partial k_z^2}{\partial \varepsilon_1}\frac{d\varepsilon_1}{dk_0^2}\right) dk_0^2\end{aligned}$$

[where the second step is based on the second of Eqs. (8)] the introduction of Eq. (43) into Eq. (40) will confirm the expected coincidence.

IV. *m* Dispersive Layers

The main advantage of Eqs. (38) to (40) over Eqs. (32) and (33) is that – apart from the common denominator $D_1 + D_2$ which plays the role of a normalizing factor – δ_i now depends formally on contributions from the i-th layer alone. (Note that the denominator $n_1^2 - n_2^2$, common to all earlier expressions for δ_i, is no longer present.) This property calls for an extension of the above analysis to an arbitrary number of layers. Our starting point is again the invariance of a determinantal equation

$$D(q_1^2, \ldots, q_m^2;\; \varepsilon_1 \ldots \varepsilon_m) = 0 \tag{44}$$

to first order in β_i which we shall assume here without proof. This means that after all layers are set into motion, each with its individual velocity β_i, and after the transverse wavenumbers and permittivities in the moving state are now denoted by $q^{(i)}$, $\varepsilon^{(i)}$, we have again

$$D(q^{(1)2}, \ldots, q^{(m)2};\; \varepsilon^{(1)} \ldots \varepsilon^{(m)}) = 0 \tag{45}$$

Denoting

$$q^{(i)2} - q_i^2 = \Delta q^{(i)2}, \quad \varepsilon^{(i)} - \varepsilon_i = \Delta\varepsilon^{(i)} \tag{46}$$

and assuming that these latter quantities are sufficiently small, we can subtract Eq. (44) from Eq. (45) and write:

$$\begin{aligned}D_1\, \Delta q^{(1)2} + \ldots + D_m\, \Delta q^{(m)2} \\ + D_{\varepsilon_i}\Delta\varepsilon^{(1)} + \ldots + D_{\varepsilon_m}\Delta\varepsilon^{(m)} = 0\,.\end{aligned} \tag{47}$$

Moreover we take Eq. (13) with $n_i^2 = \varepsilon_i$ and express the wavenumbers of the rest frame $(k_0^{(i)}, k_z^{(i)})$ in terms of laboratory wavenumbers (k_0, k_z) with the help of Equations (16). Neglecting terms of order β_i^2 we obtain

$$\varepsilon^{(i)} k_0^2 - k_z^2 - q^{(i)2} = 2(\varepsilon_i - 1)\,\beta_i k_0 k_z \qquad (48)$$

from which we subtract the corresponding relation for the quiescent state:

$$\varepsilon_i k_{00}^2 - k_{z0}^2 - q_i^2 = 0\,. \qquad (49)$$

Denoting

$$k_0^2 - k_{00}^2 = \Delta k_0^2, \qquad k_z^2 - k_{z0}^2 = \Delta k_z^2 \qquad (50)$$

and neglecting again small quantities of second order this yields

$$-\varepsilon_i \Delta k_0^2 + \Delta k_z^2 + \Delta q^{(i)2} = -2(\varepsilon_i - 1)\,\beta_i k_0 k_z + k_0^2\,\Delta\varepsilon^{(i)} \qquad (51)$$

where $i = 1, \dots, m$.

These equations together with Eq. (47) form a set of $m+1$ linear equations for the $m+2$ wavenumber increments Δk_0^2, Δk_z^2, $\Delta q^{(1)2}, \dots, \Delta q^{(m)2}$. In the straightforward extension of Eq. (29) we consider now k_z as dependent on the $2m+1$ variables k_0, $\varepsilon_1, \dots, \varepsilon_m$, $\beta_1, \dots, \beta_m$:

$$k_z = k_z(k_0; \varepsilon_1, \dots, \varepsilon_m; \beta_1, \dots, \beta_m)\,. \qquad (52)$$

Setting first all $\beta_i = 0$ we can derive expressions for the partial derivatives:

$$\partial k_z^2/\partial k_0^2 = \Sigma\, \varepsilon_\nu \mathfrak{D}_\nu / \Sigma\, \mathfrak{D}_\nu; \quad \partial k_z^2/\partial \varepsilon_i = (\mathfrak{D}_{\varepsilon_i} + k_0^2 \mathfrak{D}_i)/\Sigma\, \mathfrak{D}_\nu; \quad \Sigma = \sum_i, \quad i = 1, 2, \dots, m \qquad (53)$$

which extend those of Eqs. (37) to m layers. Our ultimate goal however, which is the calculation of Δk_z for vanishing Δk_0 can be reached directly from the set of Eqs. (47) and (51) if only $\Delta\varepsilon^{(i)}$ is expressed in terms of the Doppler shift experienced by the i-th layer and the dispersion of ε_i. From Eqs. (16) the Doppler shift is

$$[\Delta k_0^{(i)2}]_{\mathrm{DS}} = \quad k_0^{(i)} - k_0 = -\beta_i k_z$$

or

$$[\Delta k_0^{(i)2}]_{\mathrm{DS}} = -2\, k_0 k_z \beta_i \qquad (54)$$

hence

$$\Delta\varepsilon^{(i)} = (\mathrm{d}\varepsilon_1/\mathrm{d}k_0^2)\,[\Delta k_0^{(i)2}]_{\mathrm{DS}} = -2\, k_0 k_z (\mathrm{d}\varepsilon_i/\mathrm{d}k_0^2)\,\beta_i\,. \qquad (55)$$

When introducing Eq. (55) into the right-hand side of Eq. (51) we find that this latter one becomes $-2\, k_0 k_z \delta_{i\infty} \beta_i$, where $\delta_{i\infty}$ has already been defined in Equation (39). Therefore, with the help of Eq. (55) we can write the set of Eqs. (47) and (51) as

$$\begin{array}{lllll}
-\varepsilon_1 \Delta k_0^2 + \Delta k_z^2 + \Delta q^{(1)} & & & & = -2\, k_0 k_z \delta_{1\infty} \beta_1 \\
-\varepsilon_2 \Delta k_0^2 + \Delta k_z^2 & + \Delta q^{(2)} & & & = -2\, k_0 k_z \delta_{2\infty} \beta_2 \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
-\varepsilon_m \Delta k_0^2 + \Delta k_z^2 & & + & \Delta q^{(m)2} & = -2 k_0 k_z \delta_{m\infty} \beta_m \\
\qquad \mathfrak{D}_1 \Delta q^{(1)2} & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & + & \mathfrak{D}_m \Delta q^{(m)2} & = 2\, k_0 \mathrm{k}_z \sum_\gamma \mathfrak{D}_{\varepsilon_\nu} (\mathrm{d}\varepsilon_\nu/\mathrm{d}k_0^2)\,\beta_\nu\,.
\end{array} \qquad (56)$$

Equations (56) are nothing more but the full dispersion relation of the moving system in a differential form which describes but small (semirelativistic and off-resonance) deviations from the quiescent state. It links $m+2$ wavenumber increments by $m+1$ equations and therefore leaves the choice of one of these open. In our dragging problem this choice is $\Delta k_0^2 = 0$. Solving now for $\Delta k_z^2 = 2\, k_z \Delta k_z$ by evaluation of determinants leads to the final result

$$-\Delta k_z / k_0 = \sum_i \delta_i \beta_i \qquad (57)$$

where

$$\delta_i = (D_i \delta_{i\infty} + \mathfrak{D}_{\varepsilon_i} (\mathrm{d}\varepsilon_i/\mathrm{d}k_0^2) / \Sigma\, \mathfrak{D}_\nu\,. \qquad (58)$$

Although Eqs. (53) and (58) are the natural extensions of Eqs. (37) and (38) which could have been written down almost without proof, there is one difference between the two-layer and the multilayer case which may be noteworthy: for two nondispersive layers a solution (q_1, q_2) of the equation $\mathfrak{D}(q_1, q_2) = 0$ for the quiescent state will remain a solution even for the moving state. In the correspond multilayer case a solution $(q_1, q_2, \dots, q_m)$ for the quiescent state will in general no longer remain a solution for the moving state. All we can do is to keep one of the q_i's, say q_1, fixed which then imposes the condition $\mathfrak{D}(q_1, q^{(2)}, \dots, q^{(m)}) = 0$ on the rest of the transverse wavenumbers, but this condition alone no longer determines their values in any unique way.

V. Geometrical Representation (Semirelativistic)

Some of the results obtained so far can easily be vizualized by a simple transition from orthogonal (k_z, k_0)-coordinates to a system with inclined axes. Going back to Eqs. (22) we rewrite them in the form

$$k_0 = k_{00} + b\, k_{z0}; \qquad k_z = k_{z0} + b^{\wedge}\, k_{00} \tag{59}$$

where

$$b = \frac{p}{n_{12}^2} = \frac{(n_1^2-1)\,\beta_1 - (n_2^2-1)\,\beta_2}{n_1^2 - n_2^2}; \qquad b^{\wedge} = \frac{p^{\wedge}}{n_{12}^2} = \frac{n_2^2(n_1^2-1)\,\beta_1 - n_1^2(n_2^2-1)\,\beta_2}{n_1^2 - n_2^2}. \tag{60}$$

We observe that whenever both b and $b^{\wedge}$ are nondispersive Eqs. (45) can be represented by a parallel projection of the point P (k_{z0}, k_{00}) on to the inclined axes of a (k_z, k_0)-system as shown in Figure 3. We consider the cases

$$\text{A:} \quad \beta_2 = \beta_1; \qquad b^{\wedge} = b = \beta_1, \tag{61}$$

$$\text{B:} \quad n_2 = 1; \qquad b^{\wedge} = b = \beta_1, \tag{62}$$

$$\text{C:} \quad \mathrm{d}n_1/\mathrm{d}k_0 = \mathrm{d}n_2/\mathrm{d}k_0 = 0; \tag{63}$$

$b^{\wedge}$, b as given by Equation (60).

In cases A and B (which are virtually the same because a restframe exists for all moving material) Eqs. (59) reduce to the semirelativistic Doppler transformations of Equation (15). We then have $b^{\wedge} = b = \beta$ and Fig. 3 becomes a (semirelativistic) Minkovski diagram. In case C however we have the more general case shown in Figure 3. Although Eqs. (59) are still linear transformations in the mathematical sense, in a physical sense they no longer describe the same thing in a frame that moves relativ to the old one. What they describe instead is a possible way of mapping a mode (k_0, k_z) of the moving system from the mode (k_{00}, k_{z0}) of the quiescent state. In order to stress this difference in meaning let us say that Eqs. (59) describe a "transmodification".

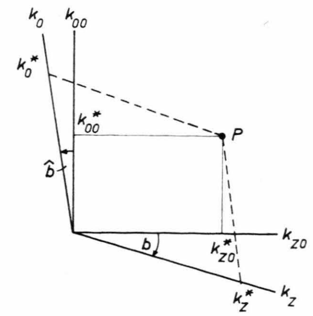

Fig. 3. Nondispersive transmodification (semirelativistic case). Angles b, $\hat{b}$ are different and exaggerated in magnitude.

There is an infinite number of transmodifications which all can translate the dispersion curve

$$\mathrm{D}_0^{(0,z)}(k_{00}, k_{z0}) = 0$$

of the quiescent system into the dispersion curve $\mathrm{D}_\beta^{(0,z)}(k_0, k_z)$ of the moving system. In order to specify a transmodification we have to give an additional condition for the wavenumbers $(k_0, k_z, q_1, \ldots, q_m)$. In this paper we have used two different conditions. One was the fixation of q_1 (which in the two-layer case automatically entrains the fixation of q_2 also). The other was the fixation of k_0 and was partly reached on a detour which consisted in a preliminary fixation of q_1, q_2 plus a differential correction thereafter. An example for case B is given in Figure 4. It shows the dispersion of the two lowest TM-modes of a waveguide which is partially filled with a cold electron plasma [4]. Such waveguides have first been investigated by Trivelpiece and Gould [5] who demonstrated already the effect of a drift velocity for slow waves.

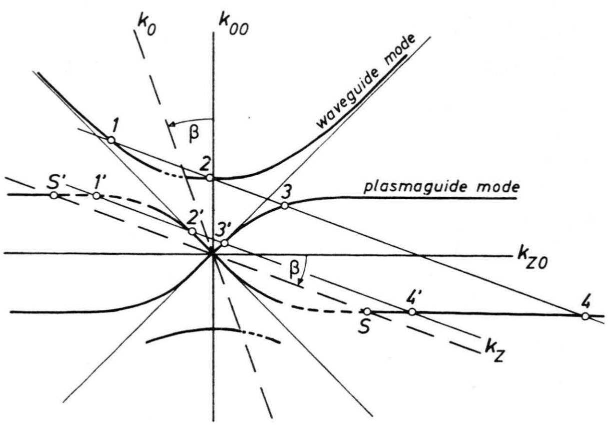

Fig. 4. Transmodification (= transformation) of the modes of a plasma-filled waveguide.

Figure 4 shows that the drift of the dielectric load destroys the symmetry which in the quiescent state has always attributed two modes with k_z opposite in sign but equal in magnitude to a particular frequency k_0. It also demonstrates the creation of a spatial cut-off (which we might call a "slow-down") at points S and S′ and of backward-wave regions (dashed parts of the curves) in either mode.

Unstable modes. In the quiescent state real values of q_i^2, ε_i, μ_i will always lead to real values of k_{00}^2 and k_{z0}^2 i. e. either to real or purely imaginary values of k_{00} and k_{z0}. This agrees with the fact that no source nor sink of energy is provided.

On the other hand Eqs. (59) show that the transmodification of a cut-off mode ($k_{z0}^2 < 0$ may render k_z, k_0 complex. In particular if we choose

$$\beta_2/\beta_1 = (n_1^2 - 1)/(n_2^2 - 1), \tag{64}$$

Equation (60) yields

$$b = 0; \quad b\hat{} = (n_1^2 - 1)\,\beta_1 . \tag{65}$$

The transmodification of any cut-off mode of the quiescent state which is characterized by

$$k_{z0} = \pm i\,|k_{z0}|; \quad k_{00} = \text{real} \tag{66}$$

yields

$$k_z = \pm i\,|k_{z0}| + (n_1^2 - 1)\,\beta_1\,k_{00}; \quad k_0 = k_{00} \tag{67}$$

which is a convectively unstable mode although the assumption $\beta_i^2 \ll 1$ (semirelativistic case) which we have used so far limits the real part of k_z as obtained from Eq. (67) to very small values.

VI. Relativistic Transmodification of $m = 0$ Modes for Two Nondispersive Media

We have shown in Appendix F that for $m = 0$ modes the transverse dispersion (determinantal equation) remains unaffected by the motion of material which by itself is non-dispersive. In this case (or any other case where a strictly invariant transverse dispersion exists) the transmodification for relativistic velocities can be obtained without difficulty.

Starting again from Eqs. (13) and (14) we have a set of four linear and two quadratic equations for the six wave vector components k_0, k_z, $k_0^{(i)}$, $k_z^{(i)}$, $i = 1, 2$. Elimination of the four components $k_0^{(i)}$, $k_z^{(i)}$ leads to quadratic equations for k_0^2 and k_z^2. Their solutions are

$$\begin{aligned} k_0^2 &= \frac{Q_0 + 2B\{(B\,Q_z - B\hat{}\,Q_0) \pm \sqrt{(B\,Q_z - B\hat{}\,Q_0)^2 + Q_0\,Q_z}\}}{1 - 4\,B\,B\hat{}} \\ k_z^2 &= \frac{Q_0 + 2B\{-(B\,Q_z - B\hat{}\,Q_0) \pm \sqrt{(B\,Q_z - B\hat{}\,Q_0)^2 + Q_0\,Q_z}\}}{1 - 4\,B\,B\hat{}} \end{aligned} \tag{68}$$

where the symbols are given by

$$\begin{aligned} &Q_0 = \Lambda^{-1}[D_2\,\bar{q}_1^2 - D_1\,\bar{q}_2^2]; && Q_z = \Lambda^{-1}[\Lambda_2\,\bar{q}_1^2 - \Lambda_1\,\bar{q}_2^2]; \; \bar{q}_i^2 = (1 - \beta_i^2)\,q_i^2, \\ &B = \Lambda^{-1}[D_2(n_1^2 - 1)\,\beta_1 - D_1(n_2^2 - 1)\,\beta_2]; && B\hat{} = \Lambda^{-1}[\Lambda_2(n_1^2 - 1)\,\beta_1 - \Lambda_1(n_2^2 - 1)\,\beta_2], \\ &\Lambda = D_2\,\Lambda_1 - D_1\,\Lambda_2; \quad D_i = 1 - n_i^2\,\beta_i^2; && \Lambda_i = n_i^2 - \beta_i^2, \quad i = 1, 2. \end{aligned} \tag{69}$$

Equations (68) and (69) now take the place of Eqs. (20) and (19) where second order terms had been neglected.

The ratio k_z/k_0 is also obtained from the solution of a quadratic equation and is given by

$$k_z/k_0 = (B\,Q_z - B\hat{}\,Q_0 \mp \sqrt{B\,Q_z - B\hat{}\,Q_0)^2 + Q_0\,Q_z})/Q . \tag{70}$$

The ambiguity in the sign of the root occurring in Eqs. (68) and (70) again corresponds to the two waves with opposite phase velocities which exist in the quiescent medium for each pair (q_1, q_2). The upper and lower signs in these equations go together.

For uniform motion ($\beta_1 = \beta_2 = \beta$) Eqs. (68) reduce to

$$\begin{aligned} k_0^2 &= (k_{00} \pm \beta\,k_{z0})^2/(1 - \beta^2); \\ k_z^2 &= (k_{z0} \pm \beta\,k_{00})^2/(1 - \beta^2). \end{aligned} \tag{71}$$

Taking the square root of Eqs. (71) we recover Eqs. (14) with (k_{00}, k_{z0}) in place of $(k_0^{(1)} = k_0^{(2)}$, $k_z^{(1)} = k_z^{(2)})$. The double sign of β reflects the two possibilities of the relative sign of $k_0^{(i)}$ and $k_z^{(i)}$ in Eqs. (13) and (14).

Thus, for uniform motion, the transmodification which leaves the transverse wavenumbers unaffected reproduces the Lorentz transformation for the wavevector.

Another special case — corresponding to that of Eq. (65) in the semirelativistic treatment — is given by the condition $B = 0$. We then have

$$\begin{aligned} B = 0: \quad & B\hat{} = -(n_i^2 - 1)\beta_i/(1 - n_i^2\,\beta_i^2), \quad i = 1, 2, \\ & k_0^2 = Q_0, \\ & k_z/k_0 = -B\hat{} \pm \sqrt{B\hat{}^2 + Q_z/Q_0}. \end{aligned} \tag{72}$$

As in Eqs. (66), (67) the transmodification of a cut-off mode which keeps k_0 real can lead to complex values of k_z for a certain range of velocities β_i which this time however is free from the restriction $\beta_i \ll 1$. We do not want to persue the relativistic case here in more detail but rather restrict ourselves to the discussion of two examples of wave dragging for nonrelativistic velocities. The second one has been submitted to verification by an experiment, the details of which will be described in a subsequent publication [6].

VII. Two Applications

Moving waveguide. For simplicity we chose a degenerate waveguide which consists of two infinite parallel conducting plates of spacing 2 a and thick enough to avoid the escape of the wave (Appendix B). The plates have a common velocity $v = \beta c$ along the z-axis and we ask for the effect of this velocity on the wavenumber k_z of a wave that propagates in the same direction. All we have to do is to apply Eq. (12) to the boundary value problem of Appendix B. We calculate the value of $\mathrm{d}q_1^2/\mathrm{d}k_0^2$ from Eqs. (12A), (13A) and the corresponding equations for the ideal conductor. The results are given in the following Table 1.

Table 1. $\delta_1 = -[\mathrm{d}q_1^2/\mathrm{d}k_0^2]$ for the parallel-plate waveguide of Appendix B.

	TE_{0n}-modes	TM_{0n}-modes
normal conductors [7] ($\omega \ll \gamma \ll \omega_p$)	$\left(\frac{\xi_n^2}{a^3}\right)(1-i)\frac{\mathrm{d}\Theta}{\mathrm{d}k_0^2} = -\left(\frac{2n-1}{4}\right)^2\left(\frac{\lambda_0}{2a}\right)^2\frac{\Theta}{a}(1-i)$	$\left(\frac{1-i}{a}\right)\left(\Theta + k_0^2\frac{\mathrm{d}\Theta}{\mathrm{d}k_0^2}\right) = (1-i)\frac{3}{4}\frac{\Theta}{a}$
ideal conductors ($\gamma = 0$, $\omega \ll \omega_p$)	0	$\frac{2c}{a\,\omega_p}$

An inspection of Table 1 leads us to the following conclusions on δ

- normal conductors give rise to real and imaginary parts of δ of equal magnitude. The imaginary part indicates a nonreciprocity in amplitude that goes along with the "drag" exerted on the phase. Thus, by moving the waveguide, depending on the direction, we can either increase or decrease the damping that exists in the quiescent state,
- the sign of δ is opposite for TE and TM modes,
- $|\delta|$ reaches the same order of magnitude for both kinds of modes since, in order to avoid cut-off, we must have

$$k_0 = 2\pi/\lambda_0 > q_1 = (2n-1)\,\pi/2a\,,$$

- δ vanishes for ideal conductors and TE modes.

Drifting electron plasma. Let us assume that a homogeneous cold and collisionfree electron plasma is drifting inside a container (glass tube) of permittivity ε_2 which itself is coated with a perfectly conducting surface. For this model of a glow discharge we assume the permittivity-velocity profile:

$$\begin{aligned} &0 \leq r < a: \quad \varepsilon = \varepsilon_1 = 1 - \omega_p^2/\omega^2, \; v = \beta c \ll c\,, \\ &a \leq r < b\,, \quad \varepsilon = \varepsilon_2\,, \qquad\qquad v = 0\,. \end{aligned} \tag{73}$$

The boundary value problem for this configuration is described in Appendix D, E and has been treated numerically in [4]. In the following we derive simple expressions for a semiquantitative treatment of the lowest waveguide modes whereas for the plasmaguide modes we refer to Ref. [4].

$m = 1$: We consider first the TE_{11}-like mode, which is the mode for which so far most experimental data [6, 8] have been collected. In order to apply Eq. (40) we use the first version of Eq. (20A) with

$$\begin{aligned} D_{\varepsilon_1} = -1/\varepsilon_2; \quad D_1 + D_2 &= \partial\eta/\partial q_1^2 + \partial\eta/\partial q_2^2 \\ &= [\mathrm{d}\eta/\mathrm{d}q_2^2]_{q_2^2 - q_1^2 = \mathrm{const}}\,. \end{aligned} \tag{74}$$

To calculate the differential on the right-hand side of Eq. (74) we need to know the dependence of η on q_1^2 and q_2^2. We use the expansion of Eq. (31A) in the geometrical limit $(b-a) \ll b$ which, when solved for η yields [recalling $s = b\,q_1$, $t = b\,q_2$ from Equation (18A)]

$$\eta = \frac{b}{b-a}\left(1 + \frac{q_2^2 - t_0^2/b^2}{q_1^2 - q_2^2}\right). \tag{75}$$

Note that when taking the differential according to Eq. (74) there is no contribution from the denominator and we have simply

$$D_1 + D_2 = \frac{b}{b-a}\,\frac{1}{q_1^2 - q_2^2} = \frac{b}{b-a}\,\frac{1}{\varepsilon_1 - \varepsilon_2}\,\frac{1}{k_0^2} \tag{76}$$

where the last part of Eq. (76) follows from the first of Equations (9). When Eqs. (76) and (74) together with ε_1 from Eq. (73) are introduced into Eq. (40) we find

$$\delta_1 = \frac{b-a}{b}\,\frac{1}{\varepsilon_2}\left(\varepsilon_2 - 1 + \frac{\omega_p^2}{\omega^2}\right)\frac{\omega_p^2}{\omega^2}\,. \tag{77}$$

It is seen from Eq. (77) that, when the plasma is surrounded by a vacuum sheath only ($\varepsilon_2 = 1$), δ rises from zero with the second power of the electron density whereas for $\varepsilon_2 > 1$ the first power dominates for small densities. This behaviour can be attributed to the circumstance that for $\varepsilon_2 > 1$ there is always an initial admixture of a TM-mode component.

$m = 0$: For TE modes $D(q_1, q_2)$ does not depend on ε_i. Since $\delta_{1\infty} = 0$, from Equation (38)

$$\delta_1 = 0 \quad \text{(TE modes)}. \tag{78}$$

For TM modes we could use a similar expansion as for $m = 1$. However from Ref. [4] we cannot derive a simple expression for $b - a \ll b$ which would correspond to the experimental situation. We may however use the approximate expression $\eta^*(s, t)$ of Eq. (27A) to obtain semiquantitative results of fair accuracy in the total range $0 < \eta < 1$. We use the first of Eqs. (9) which in terms of s^2, t^2, $\varepsilon_1 = \eta\, \varepsilon_2$, and ε_2 reads

$$t^2 - s^2 = b^2 (q_2^2 - q_1^2) = b^2 k_0^2 \varepsilon_2 (1 - \eta). \tag{79}$$

Combining Eqs. (79), (27A) and (29A) results in a quadratic equation for t^2 which when resolved for t^2 yields

$$2\, t^2 = 2\, b^2 q_2^2 = t_0^2 + t_1^2 + \vartheta\, \eta - \sqrt{(t_0^2 + t_1^2 + \vartheta\, \eta)^2 - 4\, t_0^2\, t_1^2} \tag{80}$$

where $t_1^2 = (b^2/a^2)\, x_0^2 + b^2 k_0^2 \varepsilon_2 (1 - \eta)$, $\vartheta = (t_0^2 - x_0^2)(b^2/a^2 - 1)$

and x_0, t_0 are given in Equations (25A) and (26A). The plus sign of the root has been suppressed because it belongs to the next higher mode. It is easy to obtain the differential of Eq. (74) from the above solution. For the term needed in Eq. (40) we obtain

$$1/(D_1 + D_2) = = b^{-2} [\mathrm{d}t^2/\mathrm{d}\eta]_{t^2 - s^2 = \mathrm{const}}. \tag{81}$$

From Eq. (80) the right-hand side of Eq. (81) is easy to calculate but does not reveal immediately the dependence on typical parameters. This dependence, however, can again be made transparent if we go to the limit of a thin container wall where $(b - a)/b \ll 1$. We give the result for δ in this limit. Combination of Eqs. (80), (81) and (40) yields

$$\delta = 2\, \frac{b - a}{a}\, \frac{1}{\varepsilon_2} \left[\frac{x_0^2}{b^2 k_0^2} + \varepsilon_2 - 1 + \frac{\omega_p^2}{\omega^2} \right] \frac{\omega_p^2}{\omega^2}. \tag{82}$$

Comparing Eq. (82))with Eq. (77) for $m = 1$, we find – apart from the factor 2 – the additional term $x_0^2/b^2 k_0^2 \approx 0.58\, (\lambda_0/2\, b)^2$ within the bracket, which makes δ dependent also on the vacuum wavelength-to-diameter ratio and which maintains the linear rise for small densities even in the case $\varepsilon_2 = 1$.

We have not tried to derive simple formulas for δ in the vicinity of the plasma guide resonance although this is the region where the strongest dragging is to be expected. Our treatment which is based on a linearization of the total dispersion relation cannot be expected to give reliable results whenever one of the derivatives becomes exeedingly large.

VIII. Conclusions

The relation of Fresnel and Lorentz for the dragging of light has been generalized to a theory of dragging for guided waves by a multitude of isotropic layers all differring in refractive index as well as in velocity ($\beta_i \ll c$).

The analysis is based on the invariance (to at least first order in β_i) of the determinantal equation which couples the fields between the different layers. (To our knowledge this property has not recieved much attention in the literature.) An essentially new feature arises from the consideration of nonuniform motion.

It has been demonstrated that this nonuniformity can (in principle) lead to the existence of unstable modes even for material which possesses such "banal" properties like being refractive, lossless and nondispersive. Our results have been obtained by establishing a one to one correspondence of a mode (k_0, k_z) in the moving state and another mode (k_{00}, k_{z0}) in the quiescent state which we called a "transmodification". For the case of a strictly invariant determinantal equation the transmodification for the two-layer problem is given for all velocities whereas otherwise a linearization of the entire dispersion relation has been used which is only applicable for $\beta^2 \ll 1$.

Apart from the (Lorentz-) transformation of wavevectors the only physical assumptions that enter into the analysis are isotropic propagation in the rest frame of the material and the existence of a determinantal equation (transverse dispersion) which is invariant with respect to differential material velocities. The method therefore is applicable to all waves which can match these assumptions and is not restricted to the usual types of electromagnetic waves.

Two examples of application have been used to demonstrate the theory, one of which has been submitted to experimental verificaiton which we shall deal with in a subsequent publication [6].

An extension of this theory to birefringent material would be interesting in view of its application to magnetoplasmas (diagnostics and stability).

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Appendix

The Determinantal Equation for Various Transverse Geometries

We use standard methods [9, 10] to obtain some relations which we need for the main part of this paper. We consider a cylindrical geometry of arbitrary cross section in the x-y-plane and a wave propagating along the z-axis. From Maxwell's equations we know, that the axial field components

$$F_z = f_z(x,y)\, e^{i\Phi}; \quad \Phi = k_z z - k_0 c t. \tag{1A}$$

(F standing likewise for E and/or B) must be solutions of the wave equation

$$(\nabla_t^2 - q^2)\, F = 0; \quad \nabla_t^2 = \nabla^2 - \partial^2/\partial z^2; \quad q^2 = n^2 k_0{}^2 - k_z{}^2, \quad n^2 = \mu\, \varepsilon, \tag{2A}$$

and that we can find the transverse field as follows

$$\mathbf{E}_t = i\, q^{-2} (k_z \nabla_t E_z - k_0\, \mathbf{e}_z \times \nabla_t B_z) \tag{3A}$$

$$\mathbf{B}_t/\mu = \mathbf{H}_t = i\, q^{-2} (k_z \nabla_t B_z/\mu + k_0\, \mathbf{e}_z \times \nabla_t\, \varepsilon\, E_z). \tag{4A}$$

With $\nabla_t = \partial \mathbf{e}_z/\partial x + \partial \mathbf{e}_y/\partial y$ where $\mathbf{e}_{x,y,z}$ are orthogonal unit vectors. The proper choice of f_z is dictated by the transverse geometry through the boundary conditions for the field components given by (1A), (3A) and (4A) at the conductor and at the dielectric interface. Let us first consider two-dimensional geometries characterized by $\partial/\partial y = 0$.

A. Two Dielectric Layers between Two Parallel Perfectly Conducting Planes

In the two dielectric regions we have

$$\begin{array}{ll} 0 \leqq x < a_1: & \nu = 1, \\ a_2 < x \leqq 0: & \nu = 2, \end{array} \quad \varepsilon_\nu\, \mu_\nu\, k_0{}^2 = k_z{}^2 + q_\nu{}^2,$$

and any linear combination of the two sets of fields

TE waves

$$\begin{aligned} \mu_\nu H_z = B_{z\nu} &= A_\nu \cos q_\nu (x - a_\nu)\, e^{i\Phi} \\ B_x &= -A_\nu (i\, k_z/q_\nu) \sin q_\nu (x - a_\nu)\, e^{i\Phi} \\ E_y &= A_\nu (i\, k_0/q_\nu) \sin q_\nu (x - a_\nu)\, e^{i\Phi} \end{aligned} \tag{5A}$$

TM waves

$$\begin{aligned} E_z &= C_\nu \sin q_\nu (x - a_\nu)\, e^{i\Phi} \\ \frac{D_x}{\varepsilon_\nu} = E_{x\nu} &= C_\nu (i\, k_z/q_\nu) \cos q_\nu (x - a_\nu)\, e^{i\Phi} \\ \mu_\nu H_\nu = B_{y\nu} &= C_\nu\, \mu_\nu\, \varepsilon_\nu (i\, k_0/q_\nu) \cos q_\nu (x - a_\nu)\, e^{i\Phi} \end{aligned} \tag{6A}$$

will satitsfy Eqs. (2A) to (4A) as well as the boundary conditions at the conductor. As there are no like contributions to the fields from each of the two sets (5A) and (6A) the boundary conditions at the dielectric interface can be given for each set separately and pure TE and TM modes are obtained. The continuity of tangential E and H fields result in the determinantal equations

$$D \equiv \begin{vmatrix} \dfrac{\cos q_1 a_1}{\mu_1} & \dfrac{\cos q_2 a_2}{\mu^2} \\ -\dfrac{\sin q_1 a_1}{q_1} & -\dfrac{\sin q_2 a_2}{q_2} \end{vmatrix} = 0 \quad \text{or} \quad \mu_1 \operatorname{tg}(q_1 a_1)/q_1 = \mu_2 \operatorname{tg}(q_2 a_2)/q_2 \tag{7A}$$

for the TE modes and

$$D \equiv \begin{vmatrix} -\sin q_1 a_1 & -\sin q_2 a_2 \\ \varepsilon_1 \dfrac{\cos q_1 a_1}{q_1} & \varepsilon_2 \dfrac{\cos q_2 a_2}{q_2} \end{vmatrix} = 0 \quad \text{or} \quad \varepsilon_1 \operatorname{ctg}(q_1 a_1)/q_1 = \varepsilon_2 \operatorname{ctg}(q_2 a_2)/q_2 \tag{8A}$$

for TM modes.

B. *Empty Space between Two Plane-parallel Lossy Conductors*

In the three regions we have

$$\begin{array}{llll} a<x: & \mu_2=1, \quad \varepsilon_2=1-\omega_p^2/(\omega^2-i\gamma\omega), & B_z=A_2\,e^{iq_2x}, & E_z=C_2\,e^{iq_2x}, \\ -a<x<a: & \mu_1=1, \quad \varepsilon_1=1, & B_z=A_1\sin q_1 x, & E_z=C_1\cos q_1 x, \\ x<-a: & \mu_2=1, \quad \varepsilon_2; & B_z=A_2\,e^{-iq_2x}, & E_z=C_2\,e^{-iq_2x}, \end{array}$$

where all fields still have to be multiplied by $e^{i\Phi}$ and where the conductor is represented by an electron plasma of plasma frequency ω_p and collision frequency γ. The only difference between this case and the preceding one is that we have to allow for damped waves which become purely outgoing waves for $|x|>a$. The wave numbers are now complex and the determinantal equations become

$$D\equiv\begin{vmatrix} \sin q_1 a & e^{iq_2a} \\ \dfrac{\cos q_1 a}{q_1} & \dfrac{e^{iq_2a}}{q_2} \end{vmatrix}=0 \quad\text{or}\quad \frac{\operatorname{ctg} q_1 a}{q_1 a}=\frac{i}{q_2 a} \quad \text{(TE modes)} \tag{9A}$$

$$D\equiv\begin{vmatrix} \cos q_1 a & e^{iq_2a} \\ \dfrac{-\sin q_1 a}{q_1} & \varepsilon_2\dfrac{i\,e^{iq_2a}}{q_2} \end{vmatrix}=0 \quad\text{or}\quad \frac{\operatorname{tg} q_1 a}{q_1 a}=\frac{i\,\varepsilon_2}{q_2 a} \quad \text{(TM modes)} \tag{10A}$$

for metals and microwave frequencies we have

$$\begin{array}{lll} \varepsilon_2\approx\omega_p^2/\gamma^2+i\,\omega_p^2/\gamma\,\omega & \text{for}\quad \omega_p\gg\gamma\gg\omega & \text{(normal conductor)} \\ \approx-\omega_p^2/\omega^2 & \text{for}\quad \gamma\ll\omega & \text{(ideal conductor)} \end{array}$$

since $\varepsilon_2 k_0^2 \gg k_0^2, a^{-2}, q_1^2$ we have $q_2^2 a_2^2 \approx \varepsilon_2 k_0^2 a^2 \gg 1$

and

$$\begin{aligned} 1/a\,q_2 &\approx 1/a\,k_0\,\varepsilon^{1/2}=(-\Theta/2\,a)\,(1-i) && \text{(normal conductor)} \\ &=-i\,c/\omega_p\,a && \text{(ideal conductor)}. \end{aligned} \tag{11A}$$

where $\Theta=(c/\omega_p)\sqrt{2\,\gamma/\omega}$ is the "skin depth"

The right hand side of Eq. (9A) being very small, we can expand

$$\operatorname{ctg}\xi/\xi\approx(\xi^2-\xi_n^2)/2\,\xi_n^2 \quad\text{near}\quad \xi_n=(2\,n-1)\,\pi/2 \quad\text{obtaining}$$

$$(a\,q_1)^2\approx\xi_n^2[1-(\Theta/a)\,(1-i)] \qquad \text{(TE modes)}. \tag{12A}$$

On the other hand $|\varepsilon_2/q_2|\approx|\varepsilon_2^{1/2}/k_0|$ on the right hand side of Eq. (10A) is very large and we can set $\operatorname{tg}\xi/\xi\approx-2/(\xi^2-\xi_n^2)$ obtaining

$$(a\,q_1)^2\approx\xi_n^2-a^2\,k_0^2\,(\Theta/a)\,(1-i) \qquad \text{(TM modes)}. \tag{13A}$$

Both expressions (12A) and (13A) are for normal conductors. For ideal conductors, according to Eq. (11A), $\Theta(1-i)$ has to be replaced by $2\,c/\omega_p$.

C. *Rectangular Waveguide with Two Dielectrics*

The axial field components of A. must now be amended, accounting for the finite transverse dimension b:

$$\left.\begin{aligned} e^{-i\Phi}B_{z\nu}&=A_\nu\cos k_{x\nu}(x-a_\nu)\,\cos\,(m\,\pi\,y/b), \\ e^{-i\Phi}B_{z\nu}&=C_\nu\sin k_{x\nu}(x-a_\nu)\,\sin\,(m\,\pi\,y/b), \end{aligned}\right\} \quad \nu=1,2. \tag{14A}$$

The situation is similar to A. However, when Eqs. (3A, (4A) are applied to the fields (14A) the y-dependence produces an interference of the two sets which replace Eqs. (5A) and (6A) and both sets are necessary to join the fields at the boundary. Equations (7A) and (8A) are now replaced by a 4 by 4 element determinantal equation which after evaluation yields

$$\left(\frac{m\,\pi}{b}\right)^2\frac{k_z^2}{k_0^2}\left[\frac{1}{q_1^2}-\frac{1}{q_2^2}\right]^2=-\left[\mu_1\frac{\xi_1\operatorname{tg}\xi_1}{a_1\,q_1^2}-\mu_2\frac{\xi_2\operatorname{tg}\xi_2}{a_2\,q_2^2}\right]\left[\varepsilon_1\frac{\xi_1\operatorname{ctg}\xi_1}{a_1\,q_1^2}-\varepsilon_2\frac{\xi_2\operatorname{ctg}\xi_2}{a_2\,q_2^2}\right] \tag{15A}$$

where $\xi_\nu=a_\nu\,q_\nu$.

The left hand side of Eq. (15A) can be written entirely in terms of transverse wave numbers. Using Eqs. (9) we have

$$D \propto \left(\frac{m\pi}{b}\right)^2 \left[\frac{n_1^2}{q_1^2} - \frac{n_2^2}{q_2^2}\right] \left[\frac{1}{q_1^2} - \frac{1}{q_2^2}\right] + [\mathcal{M}][\mathcal{L}] = 0 \tag{16A}$$

where the meaning of the symbols $[\mathcal{M}]$ and $[\mathcal{L}]$ is evident from comparison with Equation (15A). For $m = 0$ Eq. (15A) decays into the two Eqs. (7A) and (8A) of Appendix A.

D. Circular Waveguide with Two Dielectrics

The axial field components are:

$0 \leqq r < a$:
$$e^{-i\Phi} B_{z1} = A_1 J_m(q_1 r) \sin m\varphi,$$
$$e^{-i\Phi} E_{z1} = C_1 J_m(q_1 r) \cos m\varphi;$$

$a \leqq r < b$:
$$e^{-i\Phi} B_{z2} = A_2 (J_m(q_2 r) - \vartheta' N_m(q_2 r)) \sin m\varphi,$$
$$e^{-i\Phi} E_{z2} = C_2 (J_m(q_2 r) - \vartheta N_m(q_2 r)) \cos m\varphi$$

where
$$\vartheta = \frac{J_m(q_2 b)}{N_m(q_2 b)}, \qquad \vartheta' = \frac{J_m'(q_2 b)}{N_m'(q_2 b)}.$$

The resulting determinantal equation is again of the above type and we have

$$m^2 (k_z^2/k_0^2) \left[\frac{1}{s^2} - \frac{1}{t^2}\right]^2 = [\mathcal{M}][\mathcal{L}] \tag{17A}$$

or

$$D \propto m^2 \left[\frac{n_1^2}{s^2} - \frac{n_2^2}{t^2}\right] \left[\frac{1}{s^2} - \frac{1}{t^2}\right] - [\mathcal{M}][\mathcal{L}] = 0 \tag{18A}$$

where
$$[\mathcal{M}] = \mu_1 (\varkappa/s^2) - \mu_2 (\mu/t^2); \qquad [\mathcal{L}] = \varepsilon_1 (\varkappa/s^2) - \varepsilon_2 (\lambda/t^2)$$
$$\varkappa = a s \frac{J_m'(a s)}{J_m(a s)}; \quad a = a/b; \quad s = b q_1; \quad t = b q_2,$$
$$\lambda = a t \frac{J_m'(a t) - \vartheta N_m'(a t)}{J_m(a t) - \vartheta N_m(a t)}; \quad \mu = a t \frac{J_m'(a t) - \vartheta' N_m'(a t)}{J_m(a t) - \vartheta' N_m(a t)}.$$

As in the rectangular case this equation splits into two for $m = 0$.

$$\begin{aligned} D(q_1^2, q_2^2; \mu_1, \mu_2) &= [\mathcal{M}] = 0 \text{ for } TE_0 \text{ modes,} \\ D(q_1^2, q_2^2; \varepsilon_1, \varepsilon_2) &= [\mathcal{L}] = 0 \text{ for } TM_0 \text{ modes,} \end{aligned} \tag{19A}$$

E. Solution of the Determinantal Equation for the Geometries of Appendix C and D

The solutions of the wave Eq. (2A) are transcendental functions which make the determinantal equation always a transcendental equation that can only be solved by numerical methods. On the other hand Eqs. (16A) and (18A) are linear in ε_i or μ_i. For simplicity we assume $\mu_1 = \mu_2 = 1$. We solve for the ratio $\varepsilon_1/\varepsilon_2 = \eta$ and write the determinantal equation in one of the two following forms

$$\begin{aligned} D(q_1^2, q_2^2; \varepsilon_1, \varepsilon_2) &= \eta(q_1^2, q_2^2) - \varepsilon_1/\varepsilon_2 = 0, \\ \overline{D}(q_1^2, q_2^2; \varepsilon_1, \varepsilon_2) &= \varphi(q_1^2, q_2^2) - \varepsilon_2/\varepsilon_1 = 0 \end{aligned} \tag{20A}$$

where $\varphi = \eta^{-1}$. For η we obtain

$$\eta = \frac{q_1^2}{q_2^2} \frac{\left(\frac{m\pi}{b}\right)^2 \left(\frac{1}{q_1^2} - \frac{1}{q_2^2}\right) + \left[\frac{\xi_1 \operatorname{tg} \xi_1}{a_1 q_1^2} - \frac{\xi_2 \operatorname{tg} \xi_2}{a_2 q_2^2}\right] \frac{\xi_2 \operatorname{ctg} \xi_2}{a_2}}{\left(\frac{m\pi}{b}\right)^2 \left(\frac{1}{q_1^2} - \frac{1}{q_2^2}\right) + \left[\frac{\xi_1 \operatorname{tg} \xi_1}{a_1 q_1^2} - \frac{\xi_2 \operatorname{tg} \xi_2}{a_2 q_2^2}\right] \frac{\xi_1 \operatorname{ctg} \xi_1}{a_1}} \tag{21A}$$

for the rectangular waveguide and

$$\eta = \frac{s^2}{t^2} \frac{m^2 (1/s^2 - 1/t^2) - \lambda (\varkappa/s^2 - \mu/t^2)}{m^2 (1/s^2 - 1/t^2) - \varkappa (\varkappa/s^2 - \mu/t^2)} \tag{22A}$$

for the circular waveguide.

In the case of a cold collisionless electron plasma surrounded by vacuum Eq. (2A) leads immediately to

$$q_2^2 - q_1^2 = k_p^2 \quad (\varepsilon_1 = 1 - k_p^2/k_0^2, \quad k_p = \omega_p/c, \quad \varepsilon_2 = 1). \tag{23A}$$

In this case, for a given k_p, we can pick a pair (q_1, q_2) which satisfies (23A) and calculate ε_1 from (20A) which immediately yields k_0. Varying q_2^2 from $-\infty$ to $+\infty$, we can find the dispersion curve without the neccessity of solving a transcendental equation.

Approximations for Circular Geometry

For cases where no high degree of accuracy is demanded and the range of k_i^2 is limited to a particular mode it can be helpful to replace the right-hand side of Eq. (22A) by simpler functions.

$m=0$: For $m=0$ Eq. (22A) reduces to

$$\eta = (\lambda/t^2)\, s^2/\varkappa \tag{24A}$$

Using the definitions of $\varkappa$ and λ given below Eq. (18A) and restricting ourselves to the first radial mode only we find that

$$s^2/\varkappa^* = -2(s^2 - s_0^2)/x_0^2 \qquad \text{where} \qquad s_0 = (b/a)\, x_0, \quad J_0(x_0) = 0 \quad x_0 \approx 2,4, \tag{25A}$$

like $s^2/\varkappa$ is an even function of s that coincides with $s^2/\varkappa$ at $s = s_0$ and around $s = 0$ and may be used as an approximation in the intervall $0 \leqq s \leqq s_0$ whereas

with [11]

$$\frac{\lambda^*}{t^2} = \frac{-1}{\ln(b/a)} \frac{t^2 - t_0^2}{t_0^2} \frac{1}{t^2} \qquad t_0 \approx \frac{b\,\pi}{2(b-a)} \left[1 + .53 \left(\frac{b-a}{b+a}\right)^{1.2}\right] \tag{26A}$$

like λ/t^2 is an even function of t that coincides with λ/t^2 at $t = t_0$ and around $t = 0$. For the ratio of the two functions we use an expression of the form

$$\eta^*(s,t) = \frac{A}{t^2} \frac{t^2 - t_0^2}{t_0^2} \frac{s^2 - s_0^2}{s_0^2} \tag{27A}$$

where we shall apply different criteria for the adjustment of A, dependinig on the range of η under consideration. Setting either

$$A = A_\infty = 2(b^2/a^2)/[\ln(b/a)] \tag{28A}$$

$$\text{for} \quad \eta \gg 1, \ t^2 \ll x_0^2, t_0^2$$

or

$$A = A_1 = b^2 t_0^2 / [(b^2 - a^2)(t_0^2/x_0^2 - 1)] \tag{29A}$$

$$\text{for} \quad \eta \lesssim 1$$

we require either the right asymptotic behaviour for $t^2 \to 0$ or the right result in the limit $\varepsilon_1 = \varepsilon_2$, demanding

$$\eta^*(x_0, x_0) = \eta(x_0, x_0) = 1. \tag{30A}$$

The difference between A_∞ and A_1 will give an estimate of the quality of the approximation.

$m=1$: An inspection of Eq. (22A) shows that the structure of η is much more complex in this case than for $m=0$. From accurate numerical calculations [4] we derived empirically the coefficients for an expansion of the type:

$$t^2 = 3.4 + \varepsilon_2 (b\,k_0)^2 (1-\eta)[B_0 + B_1(1-\eta)] \tag{31A}$$

which can be used if $\varepsilon_2 (b\,k_0/\pi)^2 \ll 1$ as well as $|1-\eta| \ll 1$ is valid. [Due to Eqs. (6) and $s = b\,q_1$, $t = b\,q_2$ from Eq. (18A) the factor in front of the square bracket equals $t^2 - s^2$. It played the role of a parameter in the numerical calculations.] The values of the coefficients B_0 and B_1 given in Ref. [4] for a/b ranging from 1/3 to 9/10 reveal that in the limit $a/b \to 1$ we have

$$B_0 + B_1 \to 1, \quad B_1 \to (b-a)/b, \quad (a/b \to 1) \tag{32A}$$

and the calculations show that in this case the relaxed conditions $\varepsilon_2 (b\,k_0/\pi)^2 \lesssim 1$ and $|1-\eta| \lesssim 1$ may be used.

F. *Invariance of the Determinantal Equation with Respect to the Relative Motion of the Dielectrics*

$m=0$ modes: The fact that the $(0,z)$-components of the wave-vector do not appear in Eqs. (7A), (8A) suggests that any motion along the z-axis will leave these equations unaltered as long as μ_ν, ε_ν are non dispersive. If both dielectrics move with a common velocity the invariance is evident in the rest frame. However, if a differential velocity exists a proof is necessary. Let us consider a TE mode. In the rest frame of dielectric (ε_1, μ_1) the fields are given by Equation (5A):

$$\mu_1 H_z^{(1)} = B_z^{(1)} = A \cos[q^{(1)}(x - q_1)],$$

$$B_x^{(1)} = -A \frac{i\,k_z^{(1)}}{q^{(1)}} \sin[q^{(1)}(x - a_1)],$$

$$E_y^{(1)} = A \frac{i\,k_0}{q^{(1)}} \sin[q^{(1)}(x - a_1)]. \tag{33A}$$

The same equations hold for the rest frame of (ε_2, μ_2) after the replacement $1 \rightarrow 2$ is made in all the labelling.

We have to join the fields (tangential E, H) continuously in any system. For reason of formal symmetry we choose the lab system. In this system we use the same labelling as we did earlier. The laws of field transformation are

$$B_{z1} = B_z^{(1)}, \qquad E_{z1} = E_z^{(1)}, \tag{34A}$$

$$\begin{aligned} \mathbf{B}_{\perp 1} &= \gamma_1 (\mathbf{B}_\perp^{(1)} + \boldsymbol{\beta}_1 \times \mathbf{E}^{(1)}), \\ \mathbf{E}_{\perp 1} &= \gamma_1 (\mathbf{E}_\perp^{(1)} - \boldsymbol{\beta}_1 \times \mathbf{B}^{(1)}), \end{aligned} \tag{35A}$$

$$H_{z1} = H_z^{(1)}, \qquad D_{z1} = D_z^{(1)}, \tag{36A}$$

$$\begin{aligned} \mathbf{H}_{\perp 1} &= \gamma_1 (\mathbf{H}_\perp^{(1)} + \boldsymbol{\beta}_1 \times D^{(1)}), \\ \mathbf{D}_{\perp 1} &= \gamma_1 (D_\perp^{(1)} - \boldsymbol{\beta}_1 \times \mathbf{H}^{(1)}). \end{aligned} \tag{37A}$$

In Cartesian coordinates Eqs. (35A) and (37A) become

$$\begin{aligned} B_{x1} &= \gamma_1 (B_x^{(1)} - \beta_1 E_y^{(1)}) ; \\ B_{y1} &= \gamma_1 (B_y^{(1)} + \beta_1 E_x^{(1)}), \\ E_{x1} &= \gamma_1 (E_x^{(1)} + \beta_1 B_y^{(1)}) ; \\ E_{y1} &= \gamma_1 (E_y^{(1)} - \beta_1 B_x^{(1)}). \end{aligned} \tag{38A}$$

For the pair (H, D) analogous equations are obtained. Transforming now the fields of Eqs. (33A) due to the invariance of phase:

$$\Phi^{(1)} = \Phi^{(2)} = \Phi \tag{39A}$$

we find

$$\begin{aligned} H_{z1} &= (A_1/\mu_1) \cos q^{(1)} (x - a_1), \\ B_{x1} = \gamma_1 (B_x^{(1)} - \beta_1 E_y^{(1)}) &= -A_1 \frac{i k_z}{q^{(1)}} \sin q^{(1)} (x - a_1), \\ E_{y1} = \gamma_1 (E_y^{(1)} - \beta_1 B_x^{(1)}) &= A_1 \frac{i k_0}{q^{(1)}} \sin q^{(1)} (x - a_1). \end{aligned} \tag{40A}$$

Setting up the boundary conditions with the fields given by Eq. (40A) will lead again to the determinantal Eq. (7A) with (q_1, q_2) being replaced by $q^{(1)}$, $q^{(2)}$. In a similar way we find that for the TE modes Eq. (8A) remains unchanged, i. e. for pure modes the determinantal equation is strictly invariant. The reason is simply that the arrays

$$[E_y, 0, 0, i B_x] \quad \text{and} \quad [H_y, 0, 0, i D_x] \tag{41A}$$

transform like the four-vectors $[k_0, 0, 0, i k_z]$ and $[k_z, 0, 0, i k_0]$.

$m \neq 0$ modes: When a y-component of the wave vector or a φ-dependence exists the situation is by far less transparent. The fields in system (1) given by Eqs. (14A), (3A) and (4A) have no vanishing components in either set and therefore cannot be arranged as in (41A). When we set up the determinant for the system of equations that expresses the continuity of tangential E and H at the interface we find that on the left-hande side of Eqs. (15A) and (17A) the following replacement must be made

$$\frac{k_z^2}{k_0^2} \left[\frac{1}{q_1^2} - \frac{1}{q_2^2} \right]^2 \rightarrow \frac{1}{k_0^2} \left\{ \frac{k_z^{(1)} - n_1^2 k_0^{(1)} \beta_1}{q^{(1)2}} \gamma_1 - \frac{k_z^{(2)} - n_2^2 k_0^{(2)} \beta_2}{q^{(2)2}} \gamma_2 \right\}^2. \tag{42A}$$

For the determinantal equation to become invariant the right-hand side of (42A) would have to become independent of β_i after $k_z^{(1)}$, $k_0^{(1)}$, $k_z^{(2)}$, $k_0^{(2)}$, k_z and k_0 are all expressed in terms of $q^{(1)}$ and $q^{(2)}$ using Eqs. (13) and (14) of the main part of this paper.

It can be shown (for instance by using the extreme relativistic limit $\beta_1 \rightarrow 1$, $\beta_2 = 0$) that this in general is not the case.

On the other hand a cancellation of linear terms indicates after evaluation that the determinantal equation is still invariant to first order in β_i.

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